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Lecture 3 | Math 105 | Calculus

Limits & Continuity of Functions

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Contents

1. The Limit of Functions

1.1. Motivation

Mathematically for the concept of a Limit, we give the following example of of a function $f(x) = x^2 - x + 2$ **& we will investigate the behavior of $f(x)$ for the values of x near $x = 2$.** Thus, we need to get the **Limit** of the **function $f(x)$ as x approaches 2; i.**e. **We shall calculate some values of** $f(x) = x^2 - x + 2$ **from Left & from Right:**

(a) From **Left:** Substituting by x **values < 2**; e.g. $f(1.995) = 3.985 \approx 4$ & $f(1.999) = 3.997 \approx 4$.

(b) From **Right:** Substituting by x **values > 2**; e.g. $f(2.005) = 4.015 \approx 4$ & $f(2.001) = 4.003 \approx 4$.

From the above calculations, we can see that when x is close to 2 on either side of 2, then the value of $f(x)$ is close to 4. We express this by using the notation of Mathematical Limit as follows:

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

1.2. Definition of Limit:

Now, we come to the Mathematical Definition of Limits, as follows:

Definition 1. If $f(x)$ becomes arbitrary close to a single number L as x approaches a from either side, then:

$$\lim_{x \to a} f(x) = L$$

which reads as "the Limit of $f(x)$ as x approaches a is L"

Important Note: In Limits we are concerned not with what happens to $f(x)$ when $x = a$, but only with what happens to $f(x)$ when x is close to a.

To evaluate different types of limits for functions, we present some Rules.

1.3. Rules (properties) for evaluating Limits

Let *a* & *c* be real numbers, and let *n* be a natural number, then:

Rule 1. $\lim\limits_{x \to a} c = c$ *i.e.* the limit of a constant *c* is equal to the same *c*.

e.g. $\lim\limits_{x \to -3} 15 = 15$.

Rule 2. $\lim\limits_{x \to a} x^n = a^n$ *i.e.* the limit of monomial is just to replace x by a.

e.g. $\lim\limits_{x \to 2} x^4 = (2)^4 = 16$ & $\lim\limits_{x \to -3} x^3 = (-3)^3 = -27$.

Rule 3. $\lim\limits_{x \to a} (f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$.

i.e. The Limit is distributed on plus or minus signs.

e.g. $\lim\limits_{x \to 2} (x^5 - x^8) = \lim\limits_{x \to 2} x^5 - \lim\limits_{x \to 2} x^8 = -224$.

Rule 4. $\lim\limits_{x \to a} c\, f(x) = c \lim\limits_{x \to a} f(x)$.

e.g. $\lim\limits_{x \to -3} -2x^3 = -2 \lim\limits_{x \to -3} x^3 = 54$.

Rule 5. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} f(x) / \lim\limits_{x \to a} g(x),\ \textit{provided that}\ \lim\limits_{x \to a} g(x) \neq 0$.

Rule 6. $\lim_{x\to a} (f(x) \;.\; g(x)) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$.

Rule 7. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$. **If *n* is even, then *a* must be positive.**

Rule 8. **If $f(x)$ a polynomial function, then:** $\lim_{x\to a} f(x) = f(a)$.

i.e. **The limit of a Polynomial is just replace (substitute) *x* by *a*.**

Example 1. Find the following limit $\lim_{x\to 2} (x^2 + 2x - 3)$.

Solution.

Since $f(x)$ is a polynomial, then we find the limit of $f(x)$ by direct substitution of $x = 2$, as follows:

$$\lim_{x\to 2}(x^2 + 2x - 3) = 2^2 + 2\times 2 - 3 = 5 .$$

1.4. Other Techniques (Rules) for Evaluating Limits

Many techniques for evaluating limits are based on the following Replacement theorem, which states that if 2 functions agree at all but a single point c, then they have identical limit behavior at $x = c$.

Replacement Theorem:

Let c be a real number & let $f(x) = g(x)$ for all $x \neq c$.

If the limit of $g(x)$ exists as $x \to c$; then the limit of $f(x)$ also exists and we have: $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$.

Important Remark:

To apply the Replacement Theorem, you can use a result from Algebra which states that for a polynomial function $f(x)$,

$f(c) = 0$ if and only if $(x - c)$ is a factor of $f(x)$.

This concept is demonstrated in the following *examples* on Replacement Theorem.

1.4.1. Limit of Rational Function:

Let $f(x) = \dfrac{g(x)}{h(x)}$ be a real function, then we take the limit for $f(x)$ as follows:

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g(x)}{h(x)}, \quad then, \ we \ check \ the \ 2 \ cases:$$

$$h(a) \neq 0 \quad or \quad h(a) = 0 \quad as \quad follows:$$

Case 1 if $h(a) \neq 0$, then the limit can be obtained by direct substitution, and we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g(x)}{h(x)} = \frac{g(a)}{h(a)}$$

Case 2 if $h(a) = 0$, then factorize $g(x)$ & $h(x)$ as follows:

$$g(x) = (x-a)\ \tilde{g}(x), \quad and \quad h(x) = (x-a)\ \tilde{h}(x), \quad then:$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{(x-a)\ \tilde{g}(x)}{(x-a)\ \tilde{h}(x)} = \lim_{x\to a} \frac{\tilde{g}(x)}{\tilde{h}(x)}$$

Note. **The technique used at Case 2 is called the dividing out technique.**

We shall demonstrate the above 2 cases in the following example.

Example 2 **Find the following limits:**

(a) $\displaystyle\lim_{x\to 1} \frac{x+3}{x^2-2}$ (b) $\displaystyle\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$.

Solution

(a) If we replace x with 1, then we find that the denominator $1^2-2=-1\neq 0$, then

$$\lim_{x\to 1} \frac{x+3}{x^2-2} = \frac{1+3}{1^2-2} = \frac{4}{-1} = -4.$$

(b) If we replace x with 2, the denominator & the numerator will be equal to zero where the $(x-2)$ is a common factor.

So, for all $x\neq 2$, you can **simplify out this factor** to obtain the following:

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)} = \lim_{x\to 2} (x-1) = 1.$$

1.4.2. The Root Trick Technique

When we try to evaluate a limit were both numerator & denominator are zeros, we rewrite the fraction with a new denominator that does not include a zero in its limit. One way to do this is to divide out like factors as shown in Example 2. Another technique is to Rationalize the root, as shown in Example 3.

Example 3. Find the following limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution. Since both the numerator & the denominator are zeros. Then, we Rationalize the root by multiplying both denominator & numerator with the Conjugate of the numerator $\sqrt{x+1}+1$, as follows:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$$

Replacing x with 0, we get

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{1}{1+1} = \frac{1}{2}$$

1.5. One – Sided Limits & Limit Existence

Sometimes you can see that a limit fail to exist when a function approaches a different values from the left of c than it approaches from the right of c. This type of behavior can be described using the concept of a one-sided limit.

1.5.1. Left Sided Limits:

Let $f(x)$ be a function and c is a real number, then: If there is a real number L, such that $f(x)$ approaches L as $x \to c$ from left, we write: $\lim_{x \to c^-} f(x) = L$ & we say L is the left-sided limit of $f(x)$.

1.5.2. Right Sided Limits:

Let $f(x)$ be a function and c is a real number, then: If there is a real number K, such that $f(x)$ approaches K as $x \to c$ from right, we write: $\lim_{x \to c^+} f(x) = K$ & K is the right-sided limit of $f(x)$.

1.5.3. Existence of Limit

Definition 2. If the left sided limit is equal to the right limit both equal to L (or K), then the limit of the function $f(x)$ as $x \to a$ exists and it is equal to L. Also, if the left limit is not equal to the right limit, then the limit of $f(x)$ as $x \to a$ doesn't exist.

Example 4. Given that $f(x)$ is a piecewise-defined function;

$$f(x) = \begin{cases} 4 + x & x < 1 \\ 4x - x^2 & x > 1 \end{cases}$$

. Then find the limit of $f(x)$ as x approaches 1.

Solution. We calculate the left sided limit & the right sided limit, and we see if they are equal or not?

Left sided Limit: $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 4 + x = 4 + 1 = 5$

Right sided Limit: $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 4x - x^2 = 4 - 1 = 3$

Since, $\lim f(x) \neq \lim f(x)$ then the limit does not exist.

2. Continuity of Functions

Function is continuous on an interval if its graph on an interval can be traced using the pencil & paper without lifting the pencil from the paper.

Definition 3. A function $f(x)$ is said to be continuous at a point $x = a$ if the following 3 conditions are satisfied:

1. $f(a)$ is defined. *i.e.* a in domain of f.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Remark: If one of the 3 conditions is not satisfied for a point a, then $f(x)$ is not continuous (discontinuous) at $x = a$.

Example 5. Given the following function

$$f(x) = \begin{cases} 2, & at \quad x = 1 \\ 4x+1 & at \quad x \neq 1 \end{cases}$$

Is the function $f(x)$ **continuous** at $\boldsymbol{x = 1}$?

Solution

1st condition: $f(1) = 2$, **then the function f (1) is defined.**

2nd condition: $\lim_{x\to 1} f(x) = \lim_{x\to 1}(4x+1) = 5$. Thus the Limit at $x = 1$ exist.

3rd condition: $\lim_{x\to 1} f(x) \neq f(1)$ **. Thus the 3rd condition not satisfied.**

Finally, we conclude that $f(x)$ is not continuous at $x = 1$.

2.1 Continuity of Polynomial Functions

Definition 4. Every polynomial function is Continuous on $\boldsymbol{R}$.

i.e. the polynomial function is continuous at any real number.

Example 6. Check weather the following function:

$$f(x) = x^7 - 3x^3 + x - 3$$

is continuous or not ?

Solution. Since this function is a polynomial then:

$f(x)$ is continuous on $(-\infty, \infty)$

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2.2 Continuity of Rational Functions

Definition 5. **Every Rational Function is continuous on its Domain.**

Example 7.
Describe the interval's for which the following function is continuous: $f(x) = \dfrac{x^2 - 1}{x}$.

Solution :

Since $f(x)$ is a rational function which is continuous on its domain & since $f(x)$ is not defined at $x = 0$.

Then, the **domain of $f(x)$ is $R - \{0\}$.**

$i.e.$ $f(x)$ **is continuous on** $(-\infty, 0) \cup (0, \infty)$.

3. Problems & Solved Examples on Limits

Problem 1.

Evaluate the following limits:

(a) $$\lim_{x \to 0} \frac{(x+2)^2 - 4}{x}$$

(b) $$\lim_{x \to 2} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$$

(c) $$\lim_{x \to 7} \frac{7 - x}{49 - x^2}$$

(d) $$\lim_{t \to 3} \frac{t^3 - 27}{t^2 + 2t - 15}$$

(e) $$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Solved Examples

Example 1. Find the following limit: $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$.

Solution

Direct substitution fails since both numerator and denominator are zero when $x = 0$. We rationalize the numerator through multiplying numerator and denominator by the conjugate of the numerator which is $\sqrt{x+4}+2$ then

$$\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x} = \lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+4}+2}$$

Replacing x with 0, we get

$$\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x} = \frac{1}{2+2} = \frac{1}{4}.$$

Example 2.

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

Solution. We calculate the left sided limit & the right sided limit, then we see if they are equal or not?

Left sided Limit $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 4 - x = 4 - 1 = 3.$

Right Sided Limit $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x - x^2 = 4 - 1 = 3.$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3;$

then we conclude that the limit exist and equal 3.

Example 3

Find the limit as $x \longrightarrow 0$ from the left and the limit as $x \longrightarrow 0$ from the right for the function

$$f(x) = \frac{|2x|}{x}$$

Solution:

you know that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$

then $|2x| = 2|x| = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0. \end{cases}$

Hence ,

$$f(x) = \frac{|2x|}{x} = \begin{cases} 2, & x > 0 \\ -2, & x < 0. \end{cases}$$

Thus we have ***f*(*x*) = -2 for all** $x < 0$ & the **limit** from the **Left is:** $\lim\limits_{x \to 0^-} \frac{|2x|}{x} = -2.$

Also ***f*(*x*) =2** for all $x > 0$ & the **limit** from the **Right is:** $\lim\limits_{x \to 0^+} \frac{|2\,x|}{x} = 2.$

Thus, *f* (*x*) approaches different limits from the Left & from the Right.

Finally, the limit of the function $f(x)$ as $x \longrightarrow 0$ does not exist.

Example 4. **Given the following function** $f(x) = \dfrac{x^2 - 1}{x - 1}$.

Discuss the continuity of $f(x)$ at $x = 1$.

Solution: **Clearly, $f(x)$ is not defined at $x = 1$. Then, $f(x)$ is not continuous (discontinuous) at $x = 1$.**

Example 5. **Given the following function:** $f(x) = \begin{cases} \frac{x}{2} - 1, & x \leq 2 \\ x + 1, & x > 2 \end{cases}$.

Discuss the continuity of $f(x)$ at $x = 2$.

Solution: **We apply the 3 conditions of continuity:**

From the definition of $f(x)$, we have $f(2)=(2/2)-1=0$ *i.e.* $f(2)$ is defined.

The **right** - sided limit**:** $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+1) = 2 + 1$ **= 3.**

The **left** - sided limit: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (\frac{x}{2} - 1)$ = **2 /2 - 1 = 1 – 1 = 0.**

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, then $\lim_{x\to 2} f(x)$ **doesn't exist.**

Hence, we conclude that: Since the Limit of $f(x)$ at $x = 2$ does not exist; then $f(x)$ is not continuous at $x = 2$.

THANK YOU...